Weyl fermions on the lattice and the non-abelian gauge anomaly

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Abstract

Starting from the Ginsparg-Wilson relation, a general construction of chiral gauge theories on the lattice is described. Local and global anomalies are easily discussed in this framework and a closed expression for the effective action can be obtained. Particular attention is paid to the non-abelian gauge anomaly, which is shown to be related to a local topological field on the lattice representing the Chern character in 4+2 dimensions.

1. Introduction

In abelian chiral gauge theories the gauge anomaly is proportional to the topological charge density and its topological significance is hence relatively easy to understand. As has recently become clear [2–5], the same is true on the lattice if the lattice Dirac operator D satisfies the Ginsparg-Wilson relation [1]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D. \tag{1.1}$$

For any value of the lattice spacing a, this identity implies an exact symmetry of the fermion action, which may regarded as a lattice version of the usual chiral rotations. Moreover the axial anomaly (which coincides with the gauge anomaly in the abelian case) arises from the non-invariance of the fermion integration measure under these transformations and can be shown to be a topological field, i.e. the associated charge does not change under local deformations of the gauge field.

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This has now led to a construction of abelian chiral gauge theories on the lattice, which complies with all the basic requirements including exact gauge invariance [6]. The fermion multiplet has to be anomaly-free for this to work out, but otherwise there are no restrictions and the anomaly cancellation can be proved on the basis of the topological nature of the anomaly alone [7].

If the gauge group is not abelian, the gauge anomaly assumes a more complicated form and its topological interpretation is not immediately clear. An important clue is provided by the Stora-Zumino descent equations [8–10], which allow one to pass from the Chern character in 4+2 dimensions (an expression proportional to the third power of the gauge field tensor) via the Chern-Simons term in 4+1 dimensions to the anomaly in 4 dimensions. It is then possible to show [11] that the anomaly is related to the index theorem in 4+2 dimensions and to the existence of certain non-contractible two-spheres in the space of gauge orbits (for a review and an extensive list of references see refs. [12,13])

In this paper a general formulation of chiral gauge theories on the lattice is proposed. The basic ansatz is the same as in the case of the abelian theories considered in ref. [6], but there are some new elements which make the approach more transparent. In particular, a direct connection between the gauge anomaly and a local topological field representing the Chern character in 4+2 dimensions will be established. Apart from providing an interesting link to the earlier work on the gauge anomaly in the continuum limit, the significance of this result is that the exact cancellation of the anomaly on the lattice is reduced to a local cohomology problem which appears to be quite tractable.

2. Lattice action and chiral projectors

In the lattice theories studied in this paper the gauge field couples to a multiplet of left-handed fermions, which transform according to some unitary representation R of the gauge group G. We do not impose any restrictions on R or G at this point except that G should be a compact connected Lie group. As usual the gauge field is represented by link variables $U(x,\mu) \in G$, where x runs over all lattice points and $\mu = 0, \ldots, 3$ labels the lattice axes. The lattice is assumed to be finite with periodic boundary conditions in all directions.

As already mentioned the use of a lattice Dirac operator D satisfying the Ginsparg-Wilson relation is a key element of the present approach to chiral gauge theories. While the details of the definition of D are largely irrelevant, Neuberger's operator

[4] is an obvious choice in this context, since it is relatively simple and has all the required technical properties. In particular, the locality of the operator and the differentiability with respect to the gauge field is rigorously guaranteed if the gauge field satisfies the bound

$$\big\| 1 - R[U(p)] \big\| < \epsilon \quad \text{for all plaquettes } p, \tag{2.1}$$

where $U(p)$ denotes the product of the link variables around p and ϵ any fixed positive number less than $\frac{1}{30}$ [14].

In the following we shall take it for granted that the gauge field action restricts the functional integral to this set of fields. This can be achieved through a modified plaquette action, for example [6]. As far as the continuum limit in the weak coupling phase is concerned, lattice actions of this type should be in the same universality class as the standard Wilson action, because the bound (2.1) constrains the gauge field fluctuations at the scale of the cutoff only and does not violate any fundamental principle such as the locality or the gauge invariance of the theory.

Chiral fields may now be defined in a natural way following the steps previously described in refs. [15–17,6]. One first observes that the operator $\hat{\gamma}_5 = \gamma_5(1 - aD)$ satisfies the relations

$$(\hat{\gamma}_5)^\dagger = \hat{\gamma}_5, \qquad (\hat{\gamma}_5)^2 = 1, \qquad D\hat{\gamma}_5 = -\gamma_5 D. \tag{2.2}$$

The fermion action

$$S_{\mathrm{F}} = a^4 \sum_x \overline{\psi}(x) D \psi(x) \tag{2.3}$$

thus splits into left- and right-handed parts if the chiral projectors for fermion and anti-fermion fields are defined through

$$\hat{P}_\pm = \tfrac{1}{2}(1 \pm \hat{\gamma}_5), \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5), \tag{2.4}$$

respectively. In particular, by imposing the constraints

$$\hat{P}_-\psi = \psi, \qquad \overline{\psi}P_+ = \overline{\psi}, \tag{2.5}$$

the right-handed components are eliminated and one obtains a classical lattice theory where a multiplet of left-handed Weyl fermions couples to the gauge field in a consistent way.

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An interesting point to note here is that the space of gauge fields satisfying the bound (2.1) decomposes into disconnected topological sectors [18,19]. In the non-trivial sectors the index of the lattice Dirac operator [3,5] is in general different from zero and it turns out that the dimensions of the spaces of left-handed fermion and anti-fermion fields are then not the same. Fermion number violating processes can thus take place, exactly as expected from the semi-classical approximation in continuum chiral gauge theories [6,16].

3. Fermion integration measure

To complete the definition of the lattice theory, the functional integration measure for left-handed fermions needs to be specified. The principal difficulty here is that the constraint (2.5) depends on the gauge field. This leads to a non-trivial phase ambiguity in the measure and eventually gives rise to the gauge anomaly.

To make this clearer let us suppose that $v_j(x)$, $j = 1, 2, 3, \ldots$, is a basis of complex-valued lattice Dirac fields such that

$$\hat{P}_- v_j = v_j, \qquad (v_k, v_j) = \delta_{kj}, \tag{3.1}$$

the bracket being the obvious scalar product for such fields. The quantum field may then be expanded according to

$$\psi(x) = \sum_j v_j(x) c_j, \tag{3.2}$$

where the coefficients c_j generate a Grassmann algebra. They represent the independent degrees of freedom of the field and an integration measure for left-handed fermion fields is thus given by

$$D[\psi] = \prod_j dc_j. \tag{3.3}$$

Evidently if we pass to a different basis

$$\tilde{v}_j(x) = \sum_l v_l(x)(\mathcal{Q}^{-1})_{lj}, \qquad \tilde{c}_j = \sum_l \mathcal{Q}_{jl} c_l, \tag{3.4}$$

the measure changes by the factor $\det \mathcal{Q}$ which is a pure phase factor since the transformation matrix $\mathcal{Q}$ is unitary.

In the following two sets of basis vectors v_j and $\tilde{v}_j$ are considered to be equivalent if they are related to each other through eq. (3.4) with $\det \mathcal{Q} = 1$. Choosing a fermion integration measure amounts to specifying an equivalence class of bases. A given basis thus represents the associated measure, but it should not be confused with the measure which is a much simpler object. In particular, any two fermion measures coincide up to a gauge field dependent phase factor. The question of how to fix this phase will occupy us throughout the rest of this paper. For the time being we assume that some particular choice has been made and proceed with the definition of the theory.

In the case of the anti-fermion fields the subspace of left-handed fields is independent of the gauge field and one can take the same orthonormal basis $\bar{v}_k(x)$ for all gauge fields. The ambiguity in the integration measure

$$\mathrm{D}[\overline{\psi}] = \prod_k \mathrm{d}\bar{c}_k, \qquad \overline{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x), \tag{3.5}$$

is then only a constant phase factor. Fermion expectation values of any product $\mathcal{O}$ of the basic fields may now be defined through

$$\langle \mathcal{O} \rangle_{\mathrm{F}} = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\, \mathcal{O}\, \mathrm{e}^{-S_{\mathrm{F}}}. \tag{3.6}$$

In the non-trivial topological sectors a constant weight factor should be included in this formula [6], but for brevity this factor is omitted here since we shall almost exclusively be concerned with the vacuum sector. The fermion partition function in this sector is

$$\langle 1 \rangle_{\mathrm{F}} = \det M, \qquad M_{kj} = a^4 \sum_x \bar{v}_k(x) D v_j(x), \tag{3.7}$$

and correlation functions of products of fermion fields may be calculated as usual by applying Wick's theorem, the propagator being given by

$$\langle \psi(x)\overline{\psi}(y) \rangle_{\mathrm{F}} = \langle 1 \rangle_{\mathrm{F}} \times \hat{P}_- S(x,y) P_+, \qquad DS(x,y) = a^{-4}\delta_{xy}. \tag{3.8}$$

Full normalized expectation values are finally obtained through

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[U]\, \mathrm{e}^{-S_{\mathrm{G}}} \langle \mathcal{O} \rangle_{\mathrm{F}}, \tag{3.9}$$

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where S_{G} denotes the gauge field action, $\mathcal{Z}$ the partition function and $\mathrm{D}[U]$ the standard integration measure for gauge fields on the lattice.

4. Locality condition

We are now left with the problem to fix the phase of the fermion integration measure. Evidently this should be done in such a way that the locality of the theory is preserved and it also seems reasonable to demand that the measure is smoothly dependent on the gauge field. In this section these conditions are given a precise meaning and a few key formulae are derived which will later prove useful when we discuss the gauge anomaly.

The dependence of the fermion measure on the gauge field is best studied by considering variations †

$$\delta_\eta U(x,\mu) = a\eta_\mu(x)U(x,\mu), \qquad \eta_\mu(x) = \eta_\mu^a(x)T^a, \tag{4.1}$$

of the link field. Requiring the measure to be smooth means that in the neighbourhood of any given gauge field there exists a differentiable basis v_j of left-handed fields which represents the measure in the way explained above. The change $\delta_\eta v_j$ of the basis vectors and the linear functional

$$\mathfrak{L}_\eta = i\sum_j (v_j, \delta_\eta v_j). \tag{4.2}$$

are then well-defined. Moreover it is easy to show that $\mathfrak{L}_\eta$ transforms according to

$$\widetilde{\mathfrak{L}}_\eta = \mathfrak{L}_\eta - i\delta_\eta \ln \det \mathcal{Q} \tag{4.3}$$

under basis transformations (3.4). Equivalent bases thus yield the same linear functional and $\mathfrak{L}_\eta$ is hence a quantity associated with the measure rather than the basis vectors v_j. Roughly speaking it tells us how the phase of the measure changes when the gauge field is deformed.

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† Without loss the gauge group G may be assumed to be a subgroup of $\mathrm{U}(n)$ for some value of n. The generators T^a are then anti-hermitean matrices and the field components $\eta_\mu^a(x)$ are real

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Starting from the definition (3.7), the variation of the effective action is now easily worked out and one obtains

$$\delta_\eta \ln \det M = \text{Tr}\{\delta_\eta D \hat{P}_- D^{-1} P_+\} - i\mathfrak{L}_\eta. \tag{4.4}$$

The first term in this equation is the naively expected one while the second arises from the gauge field dependence of the measure. $\mathfrak{L}_\eta$ is hence referred to as the measure term in the following. Moreover, taking the linearity of $\mathfrak{L}_\eta$ into account, an associated current $j_\mu(x)$ may be defined through

$$\mathfrak{L}_\eta = a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x). \tag{4.5}$$

As will soon become clear this current plays a central rôle in the present approach to chiral gauge theories. In particular, we shall show in sect. 6 that the measure can be reconstructed from the current under certain conditions.

Whether a euclidean field theory is local or not is usually evident from the action. The situation here is slightly more complicated, because the fermion integration measure is not a product of local measures. An important point to note however is that the effective action is the only place where the non-trivial structure of the measure shows up. In particular, in the vacuum sector the fermion integrals $\langle \mathcal{O} \rangle_\text{F}$ are equal to the partition function $\langle 1 \rangle_\text{F}$ times a factor which is independent of the measure.

We are thus led to require that the current $j_\mu(x)$ which is induced by the fermion measure is a local expression in the gauge field. The measure term $\mathfrak{L}_\eta$ then assumes the form of a local counterterm, i.e. the interaction vertices which arise from the gauge field dependence of the fermion measure are local. The locality of the theory is thus preserved and the arbitrariness in the phase of the measure is greatly reduced.

5. Gauge anomaly

Another fundamental requirement on the fermion measure is that it should not break the gauge symmetry. In particular, the effective action should be gauge-invariant. On the lattice the group of gauge transformations is connected and it thus suffices to consider infinitesimal gauge transformations. One should not conclude from this that there are no global anomalies, but as will become clear later they arise in slightly different ways than expected from the semi-classical analysis.

Infinitesimal gauge transformations are generated by lattice fields $\omega(x)$ with values in the Lie algebra of the gauge group. The corresponding variations of the link field are obtained by substituting

$$\eta_\mu(x) = -\nabla_\mu \omega(x) \tag{5.1}$$

in eq. (4.1), where the gauge-covariant forward difference operator ∇_μ is given by

$$\nabla_\mu \omega(x) = \frac{1}{a} \left[U(x,\mu)\omega(x + a\hat{\mu})U(x,\mu)^{-1} - \omega(x) \right] \tag{5.2}$$

($\hat{\mu}$ denotes the unit vector in direction μ). Taking the transformation behaviour of the Dirac operator into account,

$$\delta_\eta D = [R(\omega), D], \tag{5.3}$$

the terms on the right-hand side of eq. (4.4) are easily worked out and for the gauge variation of the effective action the result

$$\delta_\eta \ln \det M = ia^4 \sum_x \omega^a(x) \left\{ \mathcal{A}^a(x) - [\nabla_\mu^* j_\mu]^a(x) \right\}, \tag{5.4}$$

$$\mathcal{A}^a(x) = \frac{ia}{2} \operatorname{tr}\{\gamma_5 R(T^a) D(x,x)\}, \tag{5.5}$$

is thus obtained. In these equations ∇_μ^* denotes the gauge-covariant backward difference operator and $D(x,y)$ the kernel representing the Dirac operator in position space. The trace is taken over Dirac and flavour indices only.

We now show that $\mathcal{A}(x)$ converges to the covariant gauge anomaly in the classical continuum limit. The calculation is practically the same as in the case of the axial anomaly which has been studied in detail in refs. [20–22]. One begins by representing the link field through

$$U(x,\mu) = \mathcal{P}\exp\left\{ a \int_0^1 dt\, A_\mu(x + (1-t)a\hat{\mu}) \right\}, \tag{5.6}$$

where $\mathcal{P}$ implies a path-ordered exponential and $A_\mu(x)$ is an arbitrary smooth gauge potential. Using the locality and differentiability properties of the kernel $D(x,y)$ established in ref. [14], it is then possible to derive an asymptotic expansion

$$\mathcal{A}^a(x) \underset{a\to 0}{\sim} \sum_{k=0}^{\infty} a^{k-4} \mathcal{O}_k^a(x). \tag{5.7}$$

The fields $\mathcal{O}_k(x)$ which occur in this series are traces of $R(T^a)$ times a polynomial of dimension k in $R[A_\mu(x)]$ and its derivatives. Moreover they must have the proper transformation behaviour under the symmetries of the lattice theory.

From eq. (5.5) it follows that $\mathcal{A}(x)$ is a gauge-covariant pseudo-scalar field which changes sign when the fermion representation R is replaced by its complex conjugate. Taking this into account, it is easy to convince oneself that all terms $\mathcal{O}_k(x)$ with dimension $k \leq 3$ have to be equal to zero. In the continuum limit we are then left with the term

$$\mathcal{A}^a(x) = c_1 d_R^{abc} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^b(x) F_{\rho\sigma}^c(x) + \mathrm{O}(a), \tag{5.8}$$

where $F_{\mu\nu}(x)$ is the field tensor associated with the gauge potential and

$$d_R^{abc} = 2i\,\mathrm{tr}\{R(T^a)[R(T^b)R(T^c) + R(T^c)R(T^b)]\}. \tag{5.9}$$

The constant $c_1 = -1/128\pi^2$ does not depend on the gauge group and can be calculated in the U(1) theory with a single fermion in the fundamental representation [20–25]. Since the gauge anomaly coincides with the axial anomaly in this case, the number may also be inferred from the index theorem [3,5,7].

Returning to the question posed at the beginning of this section, the results obtained above show that the effective action is gauge-invariant if (and only if)

$$[\nabla_\mu^* j_\mu]^a(x) = \mathcal{A}^a(x). \tag{5.10}$$

In other words, the phase of the fermion measure should be chosen so that the associated current satisfies this equation. Together with eq. (4.4) the gauge invariance of the effective action moreover implies that $j_\mu(x)$ has to be gauge-covariant.

As is well-known one cannot have both, locality and gauge invariance, unless the anomaly cancellation condition

$$d_R^{abc} = 0 \tag{5.11}$$

is fulfilled. There is more than one way to prove this in the present framework, a quick argument being that an expansion similar to eq. (5.7) must exist in the case of the current $j_\mu(x)$ too, since it is required to be local and smoothly dependent on the gauge field. In the continuum limit the anomaly is hence equal to the divergence of a covariant local current which is a polynomial of dimension 3 in the gauge potential and its derivatives. This is only possible if the anomaly vanishes at $a = 0$, i.e. if eq. (5.11) holds.

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6. Integrability condition

So far we have assumed that the current $j_\mu(x)$ is obtained from a given fermion measure through eqs. (4.2) and (4.5). We now show that any prescribed current satisfying a certain integrability condition arises from a measure in this way. The relation between the measure and the current is thus invertible and one may adopt the point of view that the latter is the fundamental object.

To derive the integrability condition we need to study the change of phase of the fermion measure along smooth curves

$$U_t(x, \mu), \qquad 0 \leq t \leq 1, \tag{6.1}$$

in the space of gauge fields. As discussed in sect. 4, the measure term $\mathfrak{L}_\eta$ tells us how the phase varies when the gauge field is deformed in a particular direction. The total change of phase along any given curve is thus given by the Wilson line

$$W = \exp\left\{ i \int_0^1 \mathrm{d}t \, \mathfrak{L}_\eta \right\}, \qquad a\eta_\mu(x) = \partial_t U_t(x, \mu) U_t(x, \mu)^{-1}. \tag{6.2}$$

In general this phase is non-integrable, i.e. the Wilson lines around closed curves are not necessarily equal to 1. To work this out we introduce the projector

$$P_t = \hat{P}_-\big|_{U=U_t} \tag{6.3}$$

and define a unitary operator Q_t through the differential equation

$$\partial_t Q_t = [\partial_t P_t, P_t] Q_t, \qquad Q_0 = 1. \tag{6.4}$$

It is easy to prove that

$$P_t Q_t = Q_t P_0 \tag{6.5}$$

and Q_t thus transports the projector P_t along the curve. In a few lines (appendix A) it is then possible to establish the identity

$$W = \det\{1 - P_0 + P_0 Q_1\} \tag{6.6}$$

for all closed curves. The Wilson loops are hence the same for all fermion measures. In other words, they represent a geometrical invariant of the measure.

Let us now assume that $j_\mu^a(x)$ is an arbitrary current depending smoothly on the gauge field. $\mathfrak{L}_\eta$ and the associated Wilson lines W may then be defined through eqs. (4.5) and (6.2) respectively. Evidently, for the current to arise from a fermion measure, a necessary condition is that eq. (6.6) holds for all closed loops. This is in fact also a sufficient condition for the existence of a such a measure. Moreover, in each topological sector this measure is uniquely determined, up to a constant phase factor, and smooth.

To prove these statements we consider a definite topological sector and choose an arbitrary reference field U_0 in this sector. Any other field U in the same sector can then be reached through a smooth curve U_t such that $U_1 = U$. If we define Q_t through eq. (6.4) as before, a basis of left-handed fields at the point U is given by

$$
v_j = \begin{cases} Q_1 w_1 W^{-1} & \text{if } j = 1, \\ Q_1 w_j & \text{otherwise,} \end{cases} \tag{6.7}
$$

where w_j denotes a fixed basis at the reference point and W the Wilson line (6.2) computed from the given current. This basis is path-dependent, but the associated measure is not, because any two curves U_t and $\widetilde{U}_t$ form a closed loop and the integrability condition (6.6) then implies that the unitary transformation relating the basis vectors v_j and $\tilde{v}_j$ has determinant 1. Taking this into account, it is easy to show that the fermion measure defined by the basis (6.7) has all the properties mentioned above.

The construction of chiral gauge theories on the lattice is thus reduced to the problem of finding a local current which fulfils the integrability condition (6.6) and the requirement of gauge invariance. Once this is achieved, the theory is completely specified up to a constant phase factor in each topological sector. Note that it suffices to define the current for all gauge fields satisfying the bound (2.1) since only these contribute to the functional integral.

7. Fermion determinant

Using the results obtained in the preceding section, we are now in a position to derive a closed expression for the fermion determinant in terms of the Dirac operator and the current $j_\mu(x)$. We shall then be able to make contact with Kaplan's approach to chiral gauge theories [26] and the earlier work of Alvarez-Gaumé et al. [28,29] and Ball and Osborn [30,31] on the effective action in the continuum theory.

Suppose U_0 is an arbitrary reference field in the vacuum sector and let w_j be a basis of left-handed fermion fields at this point. As explained above, the basis (6.7) then represents the fermion measure at any other point U in the vacuum sector, up to a constant phase factor. If we insert this basis in eq. (3.7), the formula

$$\det M \det M_0^\dagger = \det\{1 - P_+ + P_+ DQ_1 D_0^\dagger\} W^{-1} \tag{7.1}$$

is obtained, where D_0 and M_0 denote the Dirac operator and fermion matrix at the reference point †. All other notations are as in eq. (6.7). To fully understand this result the following remarks may be helpful.

(a) The integrability condition guarantees that the right-hand side of the equation does not depend on the curve U_t which has been chosen to connect $U = U_1$ with U_0. In other words, the path-dependence of the determinant and the Wilson line W precisely cancel each other. Note incidentally that the constant phase ambiguity of the measure drops out in the product of determinants.

(b) Using eqs. (2.2) and (6.5) it is easy to check that $DQ_1 D_0^\dagger$ commutes with γ_5. The determinant of this operator in the subspace of left-handed anti-fermion fields coincides with the determinant on the right-hand side of eq. (7.1) and the chiral nature of the expression is thus evident.

(c) In sect. 8 we shall show that the current $j_\mu(x)$ vanishes in the classical continuum limit if the fermion multiplet is anomaly-free. The Wilson line W consequently does not contribute to the fermion determinant in this limit.

(d) Concerning the operator Q_t we note that the differential equation (6.4) may be rewritten in the form

$$\partial_t Q_t = \tfrac{1}{2} a \left[\gamma_5 \partial_t D_t, P_t\right] Q_t, \qquad D_t = D|_{U=U_t}. \tag{7.2}$$

Close to the classical continuum limit, and when acting on fermion fields with frequencies far below the lattice cutoff, the operator is hence equal to 1 up to terms of order a. In particular, in eq. (7.1) the operator Q_1 only affects the contribution of the high-frequency modes and it may, therefore, be regarded as part of the lattice regularization prescription for the chiral determinant.

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† Another expression for the effective action, involving the Dirac operator and the current $j_\mu(x)$ only, may be obtained by integrating eq. (4.4) along any particular path. The idea has recently been pursued by Suzuki [33] in abelian chiral gauge theories

(e) So far the reference field U_0 has been assumed to be fixed and the factor $\det M_0^\dagger$ on the left-hand side of eq. (7.1) is then just a constant. Since U_0 and U_1 are interchangeable in this equation, another option is to interpret U_0 as a second gauge field and $\det M_0^\dagger$ as the determinant arising from a multiplet of right-handed fermions. In the present framework the formulation of such left-right symmetric chiral gauge theories thus appears to be particularly natural.

Having clarified the structure of eq. (7.1), we now briefly discuss how the formula relates to Kaplan's approach to chiral gauge theories [26]. In the version proposed by Shamir [27], this approach starts from a gauge theory in 4+1 dimensions with a multiplet of massive Dirac fermions, where the additional coordinate is assumed to range between 0 and T with Dirichlet boundary conditions on the gauge and fermion fields. The reduction to 4 dimensions is then achieved by noting that the Dirac operator admits chiral surface modes whose interactions at large T are described by an effective chiral gauge theory.

Contact with the present framework can now be made if we identify the fifth coordinate, scaled to the range $[0,1]$, with the parameter t of the path U_t. The path thus becomes a gauge field in 4+1 dimensions with boundary values U_0 and U_1. In particular, we can compare the fermion determinant in 4+1 dimensions with the determinant on the right-hand side of eq. (7.1) and it is then conceivable that they agree in the limit where the lattice spacing in the fifth dimension is sent to 0 and T to infinity. Preliminary studies suggest that this is indeed what happens if the lattice Dirac operator in 4+1 dimensions is chosen appropriately, but the details are complicated and will not be presented here. It is interesting to note, however, that from the point of view of the higher-dimensional theory, the Wilson line W in eq. (7.1) amounts to adding a local counterterm to the gauge field action. The term cancels the dependence of the fermion determinant on the gauge field in the interior of the space-time volume and thus allows one to reduce the theory to 4 dimensions, the dynamically relevant degrees of freedom being the boundary values U_0 and U_1.

In the continuum limit the phase of the fermion determinant is known to be proportional to the η-invariant of the Dirac operator in 4+1 dimensions [28–31]. One considers the massless Dirac operator in this case and uses Pauli-Villars regularization or analytic continuation methods to define the determinant, but otherwise the setup is the same as the one described above. The η-invariant, Kaplan's approach and the results obtained here are hence closely related to each other. As discussed by Kaplan and Schmaltz [32], the formula for the effective action of refs. [28–31] may in fact be directly derived from the fermion integral in 4+1 dimensions.

8. Classical continuum limit

To complete the construction of the lattice theory we still need to prove that there exists a local current $j_\mu(x)$ satisfying the requirement of gauge invariance and the integrability condition. The aim in the following lines is to determine the general solution of this problem in the classical continuum limit. Along the way an important simplification is achieved by considering the integrability condition in its differential form. Global anomalies and the relation between this equation and the gauge anomaly are further topics which will be addressed.

The differential form of the integrability condition,

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta + a\mathfrak{L}_{[\eta,\zeta]} = i\,\mathrm{Tr}\big\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\big\}, \tag{8.1}$$

is obtained by computing the variation of the Wilson loop and the determinant in eq. (6.6) under infinitesimal deformations of the loop U_t. The equation may also be derived in a more direct way, starting from the representation (4.2) of the measure term and making use of the identity †

$$\delta_\eta\delta_\zeta - \delta_\zeta\delta_\eta + a\delta_{[\eta,\zeta]} = 0 \tag{8.2}$$

and the defining properties of the basis vectors v_j.

An important point to note is that the integrability condition is a slightly stronger constraint than its differential form. To make this completely clear, let us assume that $\mathfrak{L}_\eta$ is an arbitrary linear functional satisfying eq. (8.1), for all gauge fields complying with the bound (2.1) and all vector fields $\eta_\mu(x)$ and $\zeta_\mu(x)$. The Wilson loop associated with any closed curve U_t of fields is then given by

$$W = h\det\{1 - P_0 + P_0 Q_1\}, \tag{8.3}$$

where h is invariant under continuous deformations of the curve. Evidently h is equal to 1 for all contractible loops, but in general this need not be so and eq. (6.6) thus imposes an additional constraint on $\mathfrak{L}_\eta$ if there are topologically non-trivial loops. Presumably the global anomalies discovered by Witten [34] are related to this observation since they arise from certain non-contractible loops in the space of gauge orbits [35]. Further studies are however required before a definite answer to this question can be given [36].

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† In eqs. (8.1) and (8.2) the variations η and ζ are assumed to be independent of the gauge field. Further terms proportional to $\delta_\eta\zeta$ and $\delta_\zeta\eta$ have to be included if this is not the case

In the classical continuum limit the differential form of the integrability condition reduces to a simple equation. To show this we follow the steps previously described in sect. 5, i.e. we insert the representation (5.6) for the link variables and assume that $\eta_\mu(x)$ and $\zeta_\mu(x)$ are restrictions to the lattice of some differentiable vector fields. The locality properties of the Dirac operator [14] then imply

$$i\,\mathrm{Tr}\big\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\big\} \underset{a\to 0}{\sim} \sum_{k=0}^{\infty} a^{k-4} \int \mathrm{d}^4x\,\mathcal{O}_k(x), \tag{8.4}$$

where the fields $\mathcal{O}_k(x)$ are traces of polynomials in $R[A_\mu(x)]$, $R[\eta_\mu(x)]$, $R[\zeta_\mu(x)]$ and their derivatives. Taking the symmetries of the expression into account, this leads to the result

$$i\,\mathrm{Tr}\big\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\big\} = c_2 \int \mathrm{d}^4x\,d_R^{abc}\epsilon_{\mu\nu\rho\sigma}\eta_\mu^a(x)\zeta_\nu^b(x)F_{\rho\sigma}^c(x) + \mathrm{O}(a), \tag{8.5}$$

the notations being the same as in sect. 5.

The proportionality constant c_2 in this equation is related to the coefficient c_1 of the anomaly. To work this out we consider a gauge variation $\eta_\mu(x) = -\nabla_\mu\omega(x)$ and note that

$$\delta_\eta\hat{P}_- = [R(\omega),\hat{P}_-], \qquad \hat{P}_-\delta_\zeta\hat{P}_-\hat{P}_- = 0. \tag{8.6}$$

It is then straightforward to establish the identity

$$i\,\mathrm{Tr}\big\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\big\} = -a^4 \sum_x \omega^a(x)\delta_\zeta\mathcal{A}^a(x) \tag{8.7}$$

and after substituting the asymptotic forms, eqs. (5.8) and (8.5), the relation

$$c_2 = -4c_1 = 1/32\pi^2 \tag{8.8}$$

is thus obtained. In passing we remark that the anomalous conservation law (5.10) is consistent with the integrability condition (8.1) in the sense that the combination of these equations does not lead to further constraints on the current $j_\mu(x)$ apart from the fact that it should transform covariantly under gauge transformations.

An important conclusion which can be drawn at this point is that $j_\mu(x) = 0$ is an acceptable choice of the current in the classical continuum limit if the fermion multiplet is anomaly-free. Both, the requirement of gauge invariance and the integrability condition in its differential form, are then satisfied. There is in fact no other

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sensible solution since it is impossible to construct a gauge-covariant polynomial of dimension 3 in the gauge potential $A_\mu(x)$ and its derivatives which transforms as an axial vector current. As far as the classical continuum limit is concerned, the theory is thus completely specified up to a constant phase factor in each topological sector.

9. Equivalent cohomology problem in 4+2 dimensions

Most gauge field configurations which contribute to the functional integral are not as smooth as those considered in the classical continuum limit and a general strategy to determine the current $j_\mu(x)$ thus needs to be developed if one is interested in constructing the complete theory. As a first step in this direction, we here show that the anomalous conservation law (5.10) and the integrability condition (8.1) can be mapped to a local cohomology problem whose solution is known to all orders in the lattice spacing.

To explain which type of cohomology problem we are heading to, let us consider the pure gauge theory on $\mathbb{R}^n$ with gauge group G and suppose $q(z)$ is a gauge-invariant polynomial in the gauge potential $A_\alpha(z)$ and its derivatives. Such fields are called *topological* if

$$\int \mathrm{d}^n z \, \delta q(z) = 0 \tag{9.1}$$

for any variation $\delta A_\alpha(z)$ of the gauge potential with compact support. Using the Stora-Zumino descent equations [8–10], it is possible to prove that all topological fields are of the form

$$q(z) = c(z) + \partial_\alpha k_\alpha(z), \tag{9.2}$$

where $c(z)$ is a linear combinations of Chern monomials

$$c_{\alpha_1 \ldots \alpha_{2m}} t^{a_1 \ldots a_m} F^{a_1}_{\alpha_1 \alpha_2}(z) \ldots F^{a_m}_{\alpha_{2m-1} \alpha_{2m}}(z) \tag{9.3}$$

and $k_\alpha(z)$ a gauge-invariant local current [37–39]. The tensor $c_{\alpha_1 \ldots \alpha_{2m}}$ in this expression has to be totally anti-symmetric and $t^{a_1 \ldots a_m}$ should be invariant under the adjoint action of the gauge group.

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The classification of topological fields modulo divergence terms is a particular case of a local cohomology problem in which the gauge symmetry plays an important rôle. From this point of view the Chern monomials represent the non-trivial cohomology classes. Depending on the gauge group, a basis of linearly independent Chern monomials is usually not difficult to find (see ref. [40] for example).

Returning to the lattice, our aim in the following paragraphs is to construct a topological field in 4+2 dimensions whose cohomology class is trivial if (and only if) there exists a local current $j_\mu(x)$ with the required properties. The added dimensions are continuous, i.e. we are concerned with lattice gauge fields

$$U(z, \mu) \in G, \qquad z = (x, t, s), \qquad \mu = 0, \ldots, 3, \tag{9.4}$$

which depend on two additional real coordinates t and s. We also introduce gauge potentials $A_t(z)$, $A_s(z)$ along these directions and define the associated field tensor through

$$F_{ts}(z) = \partial_t A_s(z) - \partial_s A_t(z) + [A_t(z), A_s(z)]. \tag{9.5}$$

Under arbitrary gauge transformations in 4+2 dimensions, the covariant derivative

$$D_r^A U(z, \mu) = \partial_r U(z, \mu) + A_r(z) U(z, \mu) - U(z, \mu) A_r(z + a\hat{\mu}) \tag{9.6}$$

then transforms in the same way as $U(z, \mu)$ and a similar statement also applies to the derivatives

$$D_r^A \hat{P}_- = \partial_r \hat{P}_- + [R(A_r), \hat{P}_-] \tag{9.7}$$

of the projector $\hat{P}_-$ (here and below the index r stands for t or s).

We now consider the field

$$q(z) = -i \operatorname{tr} \left\{ \left[\tfrac{1}{4} \hat{\gamma}_5 [D_t^A \hat{P}_-, D_s^A \hat{P}_-] + \tfrac{1}{4} [D_t^A \hat{P}_-, D_s^A \hat{P}_-] \hat{\gamma}_5 \right. \right.$$

$$\left. \left. + \tfrac{1}{2} R(F_{ts}) \hat{\gamma}_5 \right] (x, x) \right\}, \tag{9.8}$$

where $[\ldots](x, y)$ denotes the kernel representing the operator enclosed in the square bracket, at fixed t and s, in the same way as $D(x, y)$ represents the Dirac operator. The trace is taken over the Dirac and flavour indices only and $q(z)$ is thus a gauge-invariant local field in 4+2 dimensions. It is also not difficult to check (appendix B)

that $q(z)$ satisfies

$$a^4 \sum_x \int \mathrm{d}t\, \mathrm{d}s\, \delta q(z) = 0 \tag{9.9}$$

for all local variations of the link variables $U(z,\mu)$ and the potential $A_r(z)$, i.e. it is a topological field.

By definition $q(z)$ is in the trivial cohomology class if it is equal to the divergence of a gauge-invariant local current. We now show that this implies the existence of a local current $j_\mu(x)$ in 4 dimensions satisfying the anomalous conservation law (5.10) and the integrability condition (8.1). The converse is also true, but we shall not prove this here.

So let us suppose that

$$q(z) = \partial_\mu^* k_\mu(z) + \partial_t k_s(z) - \partial_s k_t(z), \tag{9.10}$$

where $k_\mu(z)$, $k_t(z)$ and $k_s(z)$ are gauge-invariant polynomials in

$$\partial_t^n \partial_s^m U(z,\mu), \qquad \partial_t^n \partial_s^m A_r(z), \qquad n+m \geq 1, \tag{9.11}$$

with coefficients that are local fields on the lattice depending on $U(z,\mu)$ and $A_r(z)$. In eq. (9.10) the symbol ∂_μ^* denotes the backward difference operator and the last two terms have been written in the form of a curl for reasons to become clear below.

Under scale transformations of the coordinates t and s, the monomials contributing to $k_\mu(z)$, $k_t(z)$ and $k_s(z)$ transform homogeneously if $A_r(z)$ is transformed in the usual way. It is then immediately clear that all terms on the right-hand side of eq. (9.10) with scale dimensions different from those of the left-hand side have to cancel. Without loss we may, therefore, assume that the (t,s)-dimensions of $k_\mu(z)$, $k_t(z)$ and $k_s(z)$ are $(1,1)$, $(1,0)$ and $(0,1)$ respectively. Taking the gauge symmetry into account, this implies

$$k_r(z) = a^4 \sum_y \lambda_{r,\mu}^a(w) K_{r,\mu}^a(w,z), \qquad w = (y,t,s), \tag{9.12}$$

$$a\lambda_{r,\mu}(w) = D_r^A U(w,\mu) U(w,\mu)^{-1}, \tag{9.13}$$

where $K_{r,\mu}(w,z)$ is a gauge-covariant local expression in the link variables.

After summing over all lattice points, eq. (9.10) thus assumes the form

$$a^4 \sum_y \left\{ \partial_t \left[\lambda_{s,\mu}^a(w) j_{s,\mu}^a(w) \right] - \partial_s \left[\lambda_{t,\mu}^a(w) j_{t,\mu}^a(w) \right] \right\}$$

$$= i \operatorname{Tr} \left\{ \hat{P}_- [D_t^A \hat{P}_-, D_s^A \hat{P}_-] - \tfrac{1}{2} R(F_{ts}) \hat{\gamma}_5 \right\}, \tag{9.14}$$

$$j_{r,\mu}^a(w) = a^4 \sum_x K_{r,\mu}^a(w,z). \tag{9.15}$$

Evidently $j_{r,\mu}(w)$ is a gauge-covariant local current depending on the link variables at the given values of t and s, but not on their derivatives with respect to these coordinates. Collecting all terms in eq. (9.14) proportional to $\partial_t \partial_s U(w,\mu)$, we thus conclude that

$$j_{t,\mu}(w) = j_{s,\mu}(w) \equiv j_\mu(w). \tag{9.16}$$

Note that the t,s-dependence of $j_\mu(w)$ arises through the link variables only, i.e. the current may be considered to be a local field in 4 dimensions which has been extended to 4+2 dimensions by letting the gauge field depend on t and s.

It is now straightforward to show that this current has all the required properties. The anomalous conservation law (5.10), for example, follows from eq. (9.14) by choosing the link variables to be independent of t and s. We may also set $A_r(z) = 0$ in this equation and in a few lines one then finds that the integrability condition (8.1) is fulfilled. This proves that the triviality of the cohomology class of $q(z)$ implies the existence a local current satisfying eqs. (5.10) and (8.1).

In the classical continuum limit the cohomology class of $q(z)$ is easily determined. As in the case of the anomaly discussed in sect. 5, the field may be expanded in a power series in a, the leading term being given by

$$q(z) = \tfrac{1}{6} c_1 d_R^{abc} \epsilon_{\alpha_1 \dots \alpha_6} F_{\alpha_1 \alpha_2}^a(z) F_{\alpha_3 \alpha_4}^b(z) F_{\alpha_5 \alpha_6}^c(z) + \mathrm{O}(a). \tag{9.17}$$

The obvious notations for gauge fields in $n = 6$ dimensions are being used in this equation, with space-time indices running from 0 to 5. If the fermion multiplet is anomaly-free, the expression on the right-hand side (which is equal to 2π times the Chern character [40]) vanishes and in this case $q(z)$ has trivial cohomology to lowest order in a. The same is true to any order in a, because the local fields which one generates at the higher orders of the expansion are all topological. Recalling the theorem quoted at the beginning of this section, this implies that they are of the

form (9.2) with $c(z) = 0$ since there are no Chern monomials with scale dimension greater than the space-time dimension.

The existence of a local current satisfying eqs. (5.10) and (8.1) is thus guaranteed to all orders in a. Moreover all what is needed to extend this result to any fixed value of the lattice spacing is the classification of all topological fields in 4+2 dimensions. Presumably the non-trivial cohomology classes on the lattice are in one-to-one correspondence with the linearly independent Chern monomials. At least this is so in the abelian case [7].

10. Concluding remarks

When studying chiral gauge theories one is often led to consider gauge and fermion fields in higher dimensions. It may well be that this is just a matter of mathematical convenience. On the other hand, the experience should perhaps be taken as an indication that chiral gauge theories are merely effective descriptions of the low-energy modes of a more fundamental theory in 4+1 or 4+2 dimensions. The approach of Kaplan and Shamir [26,27] provides a concrete model for this and it would be important to work out its relation to the framework presented in this paper in full detail, following the lines sketched in sect. 7.

In the continuum limit the gauge anomaly cancels if the tensor d_R^{abc} vanishes. The same is presumably true on the lattice, but a complete proof of this has only been given in abelian theories so far [6]. For non-abelian gauge groups the current status is that the anomaly cancellation has been established to all orders of an expansion in powers of the lattice spacing. Moreover, as explained in sect. 9, the problem has been reduced to classifying the topological fields in 4+2 dimensions, which does not seem to be an impossible task.

Global anomalies are a separate issue which requires control over the first homotopy group of the space of lattice gauge fields satisfying the bound (2.1). One may be able to achieve this by noting that such fields are continuous on the scale of the lattice spacing up to gauge transformations. The topology of the space of gauge orbits is hence expected to be essentially the same as in the continuum theory.

Appendix A

To prove eq. (6.6) we choose a differentiable basis v_j of left-handed fields representing the fermion measure along the curve such that $v_j|_{t=1} = v_j|_{t=0}$ (this is always possible if the measure is smooth). The measure term in eq. (6.2) is then given by

$$\mathfrak{L}_\eta = i \sum_j (v_j, \partial_t v_j). \tag{A.1}$$

Taking the properties of the operator Q_t into account, we have

$$v_j = Q_t \sum_l v_l|_{t=0} (\mathcal{S}^{-1})_{lj}, \tag{A.2}$$

where $\mathcal{S}$ is some unitary transformation matrix satisfying $\mathcal{S}|_{t=0} = 1$. When inserted in eq. (A.1) this yields

$$\mathfrak{L}_\eta = -i\partial_t \ln \det \mathcal{S} \tag{A.3}$$

and the Wilson loop W is thus equal to $\det \mathcal{S}|_{t=1}$. At $t=1$ the matrix $\mathcal{S}$ represents the action of Q_1 in the subspace of left-handed fields. In particular, its determinant coincides with the determinant of Q_1 in this subspace, i.e. with the right-hand side of eq. (6.6).

Appendix B

Starting from the definition (9.8) it is straightforward to show that

$$a^4 \sum_x q(z) = i \operatorname{Tr}\left\{ \hat{P}_- [\partial_t \hat{P}_-, \partial_s \hat{P}_-] - \tfrac{1}{2}\partial_t [R(A_s)\hat{\gamma}_5] + \tfrac{1}{2}\partial_s [R(A_t)\hat{\gamma}_5] \right\}. \tag{B.1}$$

As a consequence we have

$$a^4 \sum_x \int \mathrm{d}t\, \mathrm{d}s\, \delta q(z) = \int \mathrm{d}t\, \mathrm{d}s\, \delta\left\{ i \operatorname{Tr}\left\{ \hat{P}_- [\partial_t \hat{P}_-, \partial_s \hat{P}_-] \right\} \right\} \tag{B.2}$$

for any local deformation of the gauge field. To evaluate the right-hand side of this equation, we make use of the identity

$$\text{Tr}\big\{\delta\hat{P}_-\,\partial_t\hat{P}_-\,\partial_s\hat{P}_-\big\} = 0, \tag{B.3}$$

which may be established by inserting $(\hat{\gamma}_5)^2 = 1$ and noting that $\hat{\gamma}_5$ anti-commutes with the derivatives of the projector. One then finds that the integrand is given by

$$\partial_t\Big\{i\,\text{Tr}\big\{\hat{P}_-[\delta\hat{P}_-,\partial_s\hat{P}_-]\big\}\Big\} - \partial_s\Big\{i\,\text{Tr}\big\{\hat{P}_-[\delta\hat{P}_-,\partial_t\hat{P}_-]\big\}\Big\} \tag{B.4}$$

and after integrating over t and s one gets zero because the variation of the gauge field is compactly supported.

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